GRADE 12 EXAMINATION
NOVEMBER 2017

ADVANCED PROGRAMME MATHEMATICS: PAPER I
MODULE 1: CALCULUS AND ALGEBRA

Time: 2 hours | 200 marks

PLEASE READ THE FOLLOWING INSTRUCTIONS CAREFULLY

1. This question paper consists of 8 pages and an Information Booklet of 4 pages (i–iv). Please check that your question paper is complete.
2. Non-programmable and non-graphical calculators may be used, unless otherwise indicated.
3. All necessary calculations must be clearly shown and writing should be legible.
4. Diagrams have not been drawn to scale.
5. Round off your answers to two decimal digits, unless otherwise indicated.

QUESTION 1

1.1 (a) Solve for x if:
$(\ln x)^2 + \ln x^2 - 3 = 0$ (6)

(b) Solve for x, in terms of p and q: $e^{|x+p|} = q$ (5)

1.2 The equation of a graph is given as $y = x^2 + |2x - 3|$.

(a) Write down the y-intercept. (1)

(b) Explain why the graph has no x-intercepts. (3)

(c) Write down the coordinates of the point at which the equation of the graph is not differentiable. (2)

(d) Determine the coordinates of the stationary point. (4)

[21]

QUESTION 2

The population of a particular city, established in 1970, is growing exponentially according to the model:

$$P = Ae^{kt}$$

where P is the population in 1 000s at time t
A and k are constants.
(Note that in 1970, t = 0).
It is given that in 1975 the population was 596 000 and in 1985 it was 889 000.

2.1 Calculate the values of A and k respectively. (7)

2.2 Hence, use the model to estimate the year in which the population will have grown to 6 000 000. (3)

[10]

QUESTION 3

3.1 It is given that $px^2 + px + 1 = 0$.
Determine a real value of p such that the solutions of the equation are of the form $x = a + bi$, where a and b are rational and $b \neq 0$. (6)

3.2 The equation $x^4 - 2x^3 + px^2 - 8x + 20 = 0$ has a solution $x = 2i$.
Prove that the equation has no real solutions, and state the real value of p. (8)

3.3 Evaluate: $i + i^2 + i^3 + ... + i^{2017}$ (4)

[18]

QUESTION 4

Prove by mathematical induction that:

$$\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\left(1-\frac{1}{16}\right)..........\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n}$$

for all integer values of n, $n \geq 2$.

[12]

QUESTION 5

5.1 A function is defined as follows, where *a* and *b* are real constants:

$$f(x) = \begin{cases} 4 & \text{if} \quad x \le 1 \\ \dfrac{4}{x} & \text{if} \quad 1 < x \le 2 \\ ax + b & \text{if} \quad x > 2 \end{cases}$$

(a) Prove that f is continuous at $x = 1$ and give a reason why it is clearly not differentiable at $x = 1$. (6)

(b) Calculate *a* and *b* such that f is differentiable at $x = 2$. (8)

5.2 Consider $f(x) = \dfrac{6x^2 - x - 1}{px - 2}$.

(a) For which value(s) of *p* will $y = 2x + 1$ be an asymptote of the graph of *f?* (5)

(b) Consider the graph of f when $p = 4$.

(i) State the nature of the discontinuity of *f*. Explain your answer. (4)

(ii) Show that f is in fact a discontinuous straight line and sketch the graph. (5)

(c) Determine $f'(x)$ when $p = 3$ and show that f has two stationary points. (7)

[35]

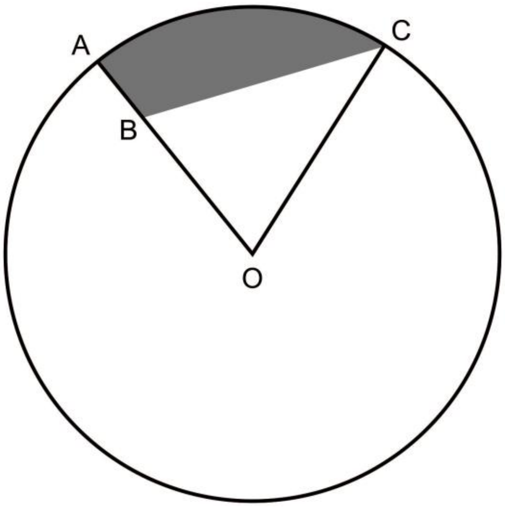
QUESTION 6

In the given diagram O is the centre of the circle and A and C lie on the circumference.

B lies on AO. AB = 2 cm, OB = 8 cm, BC = 10 cm.

6.1 Calculate the size of angle $B\hat{O}C$. (4)

6.2 Determine the area of the shaded region bounded by AB, BC and arc AC.



(6)
[10]

QUESTION 7

7.1 If $y = -\frac{1}{\sqrt{4x+3}}$ then $\frac{dy}{dx} = \frac{m}{(4x+3)^n}$.

Write down the values of *m* and *n* respectively. (5)

7.2 Given: $\sin y + \cos x = 1$ and $0 \le y \le \frac{\pi}{2}$

(a) Find $\frac{dy}{dx}$. (5)

(b) Calculate, without a calculator, the gradient of the given curve when $x = \frac{\pi}{3}$. (5)

7.3 Rajesh wants to find a solution to the equation $\tan x + x^2 + 1 = 0$, using Newton's method.

(a) If he uses $x = -1$ as an initial value, calculate Rajesh's 4^{th} iteration as it would appear on his calculator, accurate to 4 decimal places. (7)

(b) Continue the iteration to calculate the answer to 7 decimal places. (2)

[24]

QUESTION 8

Suzie is working with a function, *g*, which passes through the point (1; 4).
She differentiates the function and finds that $g'(x) = 4x^3 + 3x^2$.

8.1 Calculate the *x*-coordinates of the points of inflection and state whether each is stationary or non-stationary. (8)

8.2 Determine the algebraic expression of the function *g.* (6)

8.3 Explain why a cubic graph will always have a point of inflection, but a quartic graph, for example, may not. (3)

[17]

QUESTION 9

9.1 It is given that: $\sec^4\theta = \sec^2\theta.\tan^2\theta + \sec^2\theta$

(a) Prove the given identity, ignoring any restrictions. (4)

(b) Hence, or otherwise, determine the integral:

$\int \sec^4\theta \, d\theta$ (7)

9.2 Find the following integrals:

(a) $\int (\sin x + \cos x)^2 \, dx$ (8)

(b) $\int (x-2)\sqrt{3x^2 - 12x + 5}\, dx$ (7)

[26]

QUESTION 10

10.1 The area between the curve $y = x^2 - 4x + 8$ and the x-axis is to be approximated using a series of rectangles of width 1 unit.

Explain why the answer will be more accurate on the interval [–1; 3] than on the interval [–1; 2]. (4)

10.2 It is given that $\int_0^4 h(x)dx = 2$.

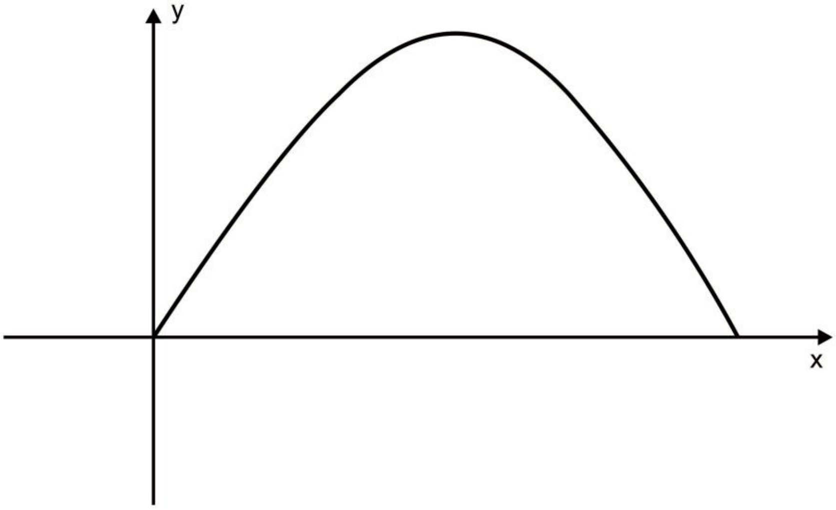
Calculate $\int_{-4}^4 h(x)dx$ if:

(a) $h(x) = h(-x)$ (2)

(b) $3h(x) = 2h(-x)$ (4)

(c) $h(x) = -h(-x)$ (2)

10.3 A parabola, passing through the origin, has a turning point at $\left(\frac{p}{2}; \frac{1}{p}\right)$.



(a) Find the equation of the graph in the form $y = a(x-b)^2 + c$ where the constants a, b and c are expressed in terms of p. (6)

(b) Prove that the area enclosed between the curve and the x-axis on the interval $0 < x < p$ is independent of p. (9)

[27]

Total: 200 marks