GRADE 12 EXAMINATION
NOVEMBER 2015

ADVANCED PROGRAMME MATHEMATICS
CORE MODULE: CALCULUS AND ALGEBRA

Time: 2 hours | 200 marks

PLEASE READ THE FOLLOWING INSTRUCTIONS CAREFULLY

1. This question paper consists of 6 pages, an Information Booklet of 2 pages (i – ii) and an Answer Sheet of 2 pages (i – ii). Please check that your question paper is complete.
2. Non-programmable and non-graphical calculators may be used, unless otherwise indicated.
3. All necessary calculations must be clearly shown and writing should be legible.
4. Diagrams have not been drawn to scale.
5. Trigonometric calculations should be done using radians and answers should be given in radians.
6. Round off your answers to two decimal digits, unless otherwise indicated.

QUESTION 1

1.1 Solve for *x*, showing your working:

(a) $e^{\pi x} - 3 = 0$ (4)

(b) $|\tan x| = 2 \qquad x \in (0;\ 2\pi)$ (6)

(c) Determine the value of n if $x = \dfrac{1}{2}$ is a solution to the equation:

$\log_{\frac{1}{2}} x + \log_2 x^n = -2$ (4)

1.2 Vusi is served a cup of Earl Grey tea at a restaurant. The temperature (P) in ºC, of the tea as it cools over time (t) in minutes can be modelled by the function:

$P = 70 \times 1{,}2^{-t} + 22$

(a) Determine:

(1) the initial temperature of the tea. (2)

(2) the room temperature. (2)

(b) Vusi can drink his tea when the temperature reaches 55 ºC but will not drink the tea if the temperature drops below 40 ºC.

Give the time interval (t) for which Vusi can drink his tea. (7)

[25]

QUESTION 2

Prove, by mathematical induction, that $5^{2n} - 1$ is divisible by 8 for every positive integer *n*.

[14]

QUESTION 3

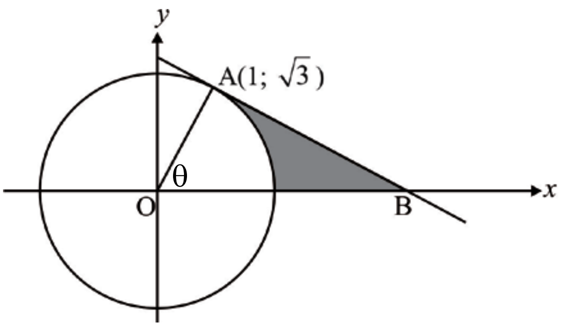
3.1 Determine, in terms of ***a*** and ***b***, the real part of the complex number:

$$\frac{a+bi}{a-bi}$$ (7)

3.2 One solution to the equation $2x^3 + px^2 + qx - 58 = 0$ is $3 - 7i$.
Calculate the values of *p* and *q* respectively. (10)

[17]

QUESTION 4



The point A(1; $\sqrt{3}$) lies on a circle with centre the origin. A tangent to the circle is drawn at A and this intercepts the x-axis at B. $A\hat{O}B = \theta$.

4.1 Calculate θ and the length of the radius of the circle. (6)

4.2 Hence, calculate the area enclosed between the circle, the tangent and the x-axis, indicated by the shaded region in the diagram. (9)

[15]

QUESTION 5

The functions p, q and r are defined, for $x \in R$, by:

$$p(x) = \frac{1}{2+3x};\quad x \neq -\frac{2}{3} \qquad q(x) = x^2 + 6 \qquad r(x) = 2x - 5$$

5.1 Find:

(a) $p'(x)$ (4)

(b) $p^{-1}(x)$ (4)

5.2 Express the following as composite functions of p, q and/or r:

(a) $\dfrac{1}{2+3\left(x^2+6\right)}$ (3)

(b) $\dfrac{2}{2+3x} - 5$ (3)

5.3 Given that $g(x) = \dfrac{q(x)}{r(x)}$, determine the coordinates of the stationary points of g. (8)

[22]

QUESTION 6

6.1 It is given that: $f(x)=\dfrac{2x^2+x-1}{x^2+px+4}$

(a) For which value(s) of p will the graph of f have:

(i) one x-intercept (4)

(ii) one vertical asymptote (3)

(b) Given that: $p=-5$

(i) Solve for x if $f\left(x\right)\geq 0$ (7)

(ii) Sketch the graph of f, including intercepts with the axes and asymptotes on the Answer Sheet. Stationary points are not required. (10)

6.2 It is given that: $g\left(x\right)=\begin{cases}-x^2-x+3 & if \quad x<0\\ \left|x-3\right| & if \quad x\geq 0\end{cases}$

(a) Showing working, with correct notation, prove that g is differentiable at $x=0$. (10)

(b) Sketch g, showing the stationary point and intercepts with the axes on the Answer Sheet. (8)

[42]

QUESTION 7

7.1 It is required to find the smallest positive solution to the equation:

$$3\left(x-2\right)^2-1=\frac{4}{x}$$

(a) Show that a solution exists on the domain $x\in\left(2;\,3\right)$. (3)

(b) Hence, find this solution using Newton's method, correct to 6 decimal places. (7)

7.2 Given: $\sin y=x$ and $0<y<\dfrac{\pi}{2}$.

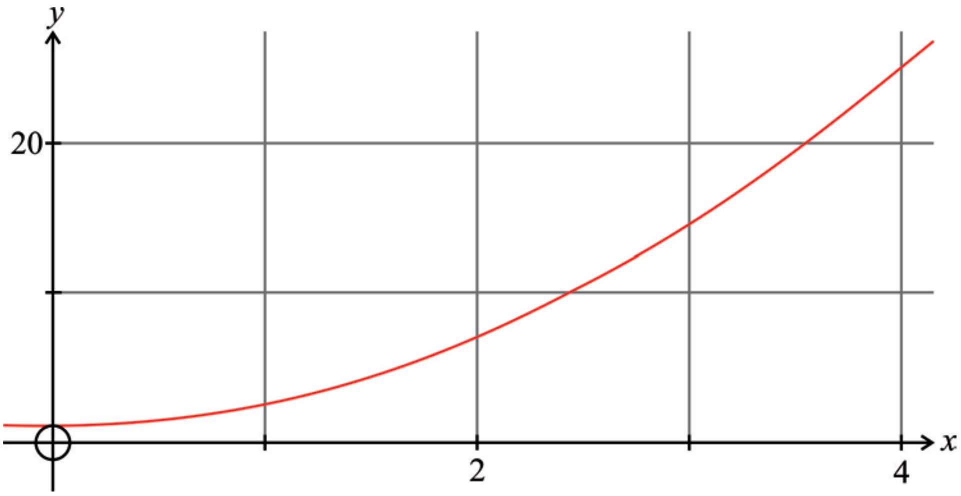
(a) Find an expression for $\dfrac{dy}{dx}$. (5)

(b) Hence, evaluate $\dfrac{dy}{dx}$ when $x=0{,}5$. (4)

[19]

QUESTION 8

Vanessa finds the area enclosed by the curve $y = qx^2 + 1$ and the x-axis, between $x = 0$ and $x = 4$, using Riemann Sums.



She finds that if she uses n rectangles, an expression for the area simplifies to:

$$A = 36 + \frac{48}{n} + \frac{16}{n^2}$$

8.1 Find the exact area. (2)

8.2 Calculate the percentage error if 16 rectangles are used. (4)

8.3 Determine, by any method, the value of q. (7)

[13]

QUESTION 9

9.1 Integrate by parts: $\int x(2x-1)^{\frac{2}{3}}\,dx$ (8)

9.2 Determine the integral:

$$\int \frac{\left(\sqrt{x}+1\right)^3}{\sqrt{x}}dx$$ (7)

9.3 An identity is given: $(\sec x - \tan x)^2 = a\sec^2 x + b\sec x\tan x + c$

(a) Determine the values of a, b and c. (5)

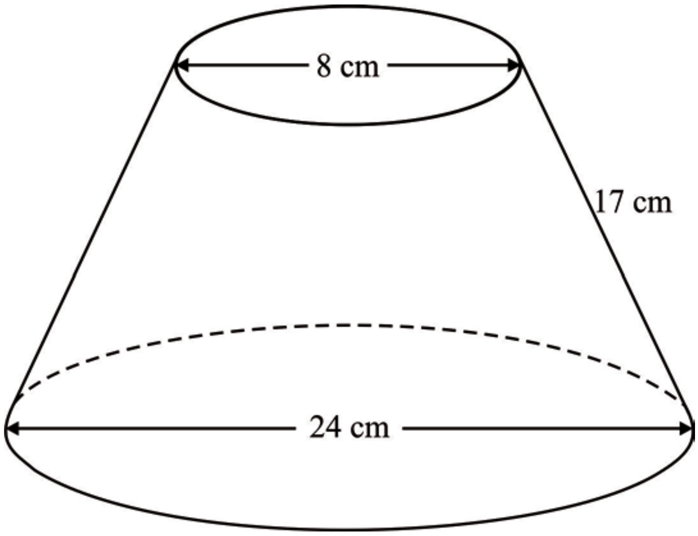
(b) Hence, determine the integral:

$$\int (\sec x - \tan x)^2\ dx$$ (5)

[25]

QUESTION 10

The diagram shows a frustum of a cone with base diameter 24 cm, top diameter 8 cm and slant height 17 cm. This has been created by rotating a certain function about the *x*-axis.



With the help of a suitable sketch on a Cartesian plane, write down the integral that would represent the volume of the frustum. You do not need to evaluate the integral.

[8]

Total: 200 marks