GRADE 12 EXAMINATION
NOVEMBER 2018

ADVANCED PROGRAMME MATHEMATICS: PAPER I
MODULE 1: CALCULUS AND ALGEBRA

MARKING GUIDELINES

Time: 2 hours | 200 marks

These marking guidelines are prepared for use by examiners and sub-examiners, all of whom are required to attend a standardisation meeting to ensure that the guidelines are consistently interpreted and applied in the marking of candidates' scripts.

The IEB will not enter into any discussions or correspondence about any marking guidelines. It is acknowledged that there may be different views about some matters of emphasis or detail in the guidelines. It is also recognised that, without the benefit of attendance at a standardisation meeting, there may be different interpretations of the application of the marking guidelines.

QUESTION 1

1.1 (a)

$$\left|x^2+x\right|=-2x-2$$
$$\therefore x^2+x=-2x-2 \text{ or } -x^2-x=-2x-2$$
$$\therefore x^2+3x+2=0 \text{ or } x^2-x-2=0$$
$$\therefore x=-1 \text{ or } -2 \text{ or } x=-1 \text{ or } 2$$
$$\textit{but a check reveals } x=-1 \text{ or } -2$$

(b)

$$\ln x^3+2\ln x^2=7$$
$$\therefore \ln x^3+\ln x^4=7$$
$$\therefore \ln x^7=7 \ (\textit{can go straight to the answer from here})$$
$$\therefore 7\ln x=7$$
$$\therefore \ln x=1$$
$$\therefore x=e$$

1.2 (a)

$$y=y_0e^{-kt}$$
$$\therefore e^{-kt}=\frac{y}{y_0}$$
$$\therefore -kt=\ln\frac{y}{y_0}$$
$$\therefore k=\frac{\ln\frac{y}{y_0}}{-t}$$

(b)

$$k=\frac{\ln\frac{0,5y_0}{y_0}}{-5700}\approx 1,216\times 10^{-4}$$

(c)

$$0,9y_0=y_0e^{-kt}$$
$$\therefore -kt=\ln 0,9$$
$$\therefore t=\frac{\ln 0,9}{-k}$$
$$\therefore t\approx 866 \textit{ years}$$

QUESTION 2

2.1 *if* $3+2i$ *is a root then so is* $3-2i$

so our equation is:

$$(x+3)\big(x-(3+2i)\big)\big(x-(3-2i)\big)=0$$

$$\therefore (x+3)\big((x-3)-2i\big)\big((x-3)+2i\big)=0$$

$$\therefore (x+3)\left((x-3)^2-4i^2\right)=0$$

$$\therefore (x+3)\left(x^2-6x+13\right)=0$$

$$\therefore x^3-3x^2-5x+39=0$$

2.2 A cubic equation will have three roots. Complex roots of polynomials with real coefficients occur in conjugate pairs so there must be at least one real root.

2.3
$$\frac{a+bi}{-b+ai}\times\frac{-b-ai}{-b-ai}$$

$$=\frac{-ab-a^2i-b^2i-abi^2}{b^2-a^2i^2}$$

$$=\frac{ab-ab-i\left(a^2+b^2\right)}{b^2+a^2}$$

$$=\frac{-i\left(a^2+b^2\right)}{b^2+a^2}$$

$$=-i$$

QUESTION 3

if $n = 1$ *we have* $2^3 - 3 = 5$ *which is clearly divisible by* 5.

Assume true for $n = k$ *viz. that*

$2^{3k} - 3^k = 5p$ *where* $p \in \mathbb{N}$ $(*)$

Now if $n = k + 1$ *we have*:

$2^{3(k+1)} - 3^{k+1}$

$= 2^{3k+3} - 3^{k+1}$

$= 2^{3k} \times 2^3 - 3 \times 3^k$

$= 8 \times 2^{3k} - 3 \times 3^k$

from $(*)$ *we have* $2^{3k} = 5p + 3^k$ *so if* $n = k + 1$ *we have*

$= 8(5p) + 8(3^k) - 3 \times 3^k$

$= 5(8p) + 5(3^k)$

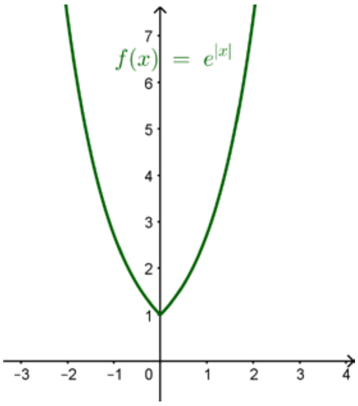
$= 5(8p + 3^k)$

which is clearly divisible by 5 ✓a

so, by the Principle of Mathematical Induction we have proved the result for $n \in \mathbb{N}$

QUESTION 4

4.1 (a)



(b) $x = 0$

4.2 *if f is differentiable at* $x = 2$

it must be continuous at $x = 2$

so $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x)$

so $2a - b - 1 = 4b - 2a + 5$ *or* $4a - 5b = 6 \quad (1)$

but also, $\lim_{x\to 2^-} f'(x) = \lim_{x\to 2^+} f'(x)$

$a = 4b - a$ *or* $a = 2b \quad (2)$

solving (1) *and* (2) *simultaneously*

$8b - 5b = 6$ *so* $b = 2$ *and* $a = 4$

QUESTION 5

5.1 *segment* $=$ *sector* $-\Delta$

$\therefore 308 = \frac{1}{2}(18^2)\theta - \frac{1}{2}(18^2)\sin\theta$

$\therefore 162\theta - 162\sin\theta - 308 = 0$

5.2 $f(\theta) = 162\theta - 162\sin\theta - 308$

$\therefore \theta_{n+1} = \theta_n - \frac{162\theta - 162\sin\theta - 308}{162 - 162\cos\theta}$

$\theta = 2{,}49984$

QUESTION 6

6.1 $f(0)=\frac{1}{2}, \quad so \quad y\text{-}int \quad \left(0;\frac{1}{2}\right)$

$\frac{2x^2-3x-2}{x-4}=0$

$\therefore 2x^2-3x-2=0$

$\therefore (2x+1)(x-2)=0$

$\therefore x\text{-}ints \quad \left(-\frac{1}{2};0\right) \quad and \quad (2;0)$

6.2 $vertical\ asymptote: x=4$

$2x^2-3x-2=(x-4)(2x+5)+R$

$so,\ oblique\ asymptote\ is\ y=2x+5$

6.3 $f(x)=\frac{2x^2-3x-2}{x-4}$

$\therefore f'(x)=\frac{(4x-3)(x-4)-1(2x^2-3x-2)}{(x-4)^2}=0$

$\therefore 4x^2-19x+12-2x^2+3x+2=0$

$\therefore 2x^2-16x+14=0$

$\therefore x^2-8x+7=0$

$\therefore (x-1)(x-7)=0$

$\therefore x=1 \quad or \quad 7$

$\therefore (1;1)\ and\ (7;25)\ are\ stationary\ points$

6.4 $f''(1)<0 \quad so \quad (1;1) \quad is\ a\ local\ maximum$

$f''(7)>0 \quad so \quad (7;25) \quad is\ a\ local\ minimum$

QUESTION 7

7.1 $x^2 + xy + y^2 = 1$

$\therefore 2x + y + x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$

$\therefore \frac{dy}{dx}\left(x + 2y\right) = -2x - y$

$\therefore \frac{dy}{dx} = \frac{-2x - y}{x + 2y}$

7.2 $At \quad A, \quad y = 0 \quad \therefore \quad x = 1$

$so, \quad at \quad A, \quad \frac{dy}{dx} = \frac{-2}{1} = -2$

$\therefore y = -2\left(x - 1\right)$

$\therefore y = -2x + 2$

QUESTION 8

8.1 $\cos\theta = \dfrac{FC}{CD}$

$\therefore FC = 0,4\cos\theta$

$\therefore A = \Delta CDF + \Delta BEG + BCFG$

$\therefore A = 2\left(\dfrac{1}{2}\times 0,4\times 0,4\cos\theta\sin\theta\right) + 0,4\times 0,4\cos\theta \quad (\Delta CDF = \Delta BEG)$

$\therefore A = 0,16\sin\theta\cos\theta + 0,16\cos\theta$

$\therefore A = 0,08\sin 2\theta + 0,16\cos\theta$

$\therefore V = 20(0,08\sin 2\theta + 0,16\cos\theta)$

$\therefore V = 1,6\sin 2\theta + 3,2\cos\theta$

8.2 $V = 1,6\sin 2\theta + 3,2\cos\theta$

$\therefore \dfrac{dV}{d\theta} = 3,2\cos 2\theta - 3,2\sin\theta = 0$

$\therefore \cos 2\theta = \sin\theta$

$\therefore \cos 2\theta = \cos\left(\dfrac{\pi}{2} - \theta\right)$

$\therefore 2\theta = \dfrac{\pi}{2} - \theta$

$\therefore 3\theta = \dfrac{\pi}{2}$

$\therefore \theta = \dfrac{\pi}{6}$

QUESTION 9

9.1 (a) $\sin^3\theta = \sin\theta \times \sin^2\theta$

$= \sin\theta\left(1-\cos^2\theta\right)$

$= \sin\theta - \sin\theta\cos^2\theta$ *as required*

(b) $\int \sin^3\theta\, d\theta = \int \sin\theta\, d\theta - \int \sin\theta\cos^2\theta\, d\theta$

$= -\cos\theta + \dfrac{\cos^3\theta}{3} + c$

9.2 **METHOD 1**

$$\int \frac{x}{\sqrt{2+x}}\,dx$$

let $u = 2 + x$ *then* $x = u - 2$ *and* $du = dx$

$$\therefore \int \frac{u-2}{u^{\frac{1}{2}}}\,du$$

$$= \int u^{\frac{1}{2}} - 2u^{-\frac{1}{2}}\,du$$

$$= \frac{2}{3}u^{\frac{3}{2}} - 4u^{\frac{1}{2}} + c$$

$$= \frac{2}{3}(2+x)^{\frac{3}{2}} - 4(2+x)^{\frac{1}{2}} + c$$

ALTERNATIVE 1

$$\int \frac{x}{\sqrt{2+x}}\,dx$$

$$= \int x(2+x)^{-\frac{1}{2}}\,dx$$

by parts $f = x$ *and* $g' = (2+x)^{-\frac{1}{2}}$

so $f' = 1$ *and* $g = 2(2+x)^{\frac{1}{2}}$

$$= 2x(2+x)^{\frac{1}{2}} - \int 2(2+x)^{\frac{1}{2}}\,dx$$

$$= 2x(2+x)^{\frac{1}{2}} - \frac{4(2+x)^{\frac{3}{2}}}{3} + c$$

QUESTION 10

10.1 $Area = \frac{10}{3} + \frac{3}{2(4)} + \frac{1}{6(4^2)}$

$= \frac{119}{32}$ (3)

10.2 It will be an over-approximation. As *n* gets larger the answer decreases.

10.3 $Area = \lim_{n \to \infty} \left(\frac{10}{3} + \frac{3}{2n} + \frac{1}{6n^2} \right)$

$= \frac{10}{3} \quad units^2$

10.4 $\frac{10}{3} \quad units^2$ since this is simply a reflection of the shaded area in the *y*-axis.

QUESTION 11

$$Area = \int_1^7 g(x) - f(x)\, dx$$

$$= \int_1^7 f(x) + kx + 1 - f(x)\, dx$$

$$= \int_1^7 kx + 1\, dx$$

$$= \left[\frac{kx^2}{2} + x \right]_1^7$$

$$so, \quad \frac{49k}{2} + 7 - \frac{k}{2} - 1 = 54$$

$$so, \quad 49k + 14 - k - 2 = 108$$

$$so, \quad 48k = 96$$

$$so \quad k = 2$$

QUESTION 12

$$vol = \pi \int_a^b \left[f(x) \right]^2 dx$$

$$\therefore 175 = \pi \int_a^b -x^2 + 6x + 4\, dx$$

$$\therefore 175 = \pi \left[-\frac{x^3}{3} + 3x^2 + 4x \right]_0^h$$

$$\therefore 175 = \pi \left(-\frac{h^3}{3} + 3h^2 + 4h \right)$$

$$\therefore -h^3 + 9h^2 + 12h - \frac{525}{\pi} = 0$$

$$\therefore x = 5{,}28\ cm$$

Total: 200 marks